**Indian Statistical Institute, Bangalore**
M. Math.
First Year, First Semester
Measure theoretic probability

Final Examination | Date: Nov. 30, 2011
Maximum marks: 100 | Duration: 3 hours

(1) Fix a natural number $n$. Let $f : I\!N \to \{0, 1, 2, \ldots, (n-1)\}$ be the function such that $i \equiv f(i) (\text{mod } n)$, in other words $f(i)$ is the reminder when $i$ is divided by $n$. Let $\mathcal{G}_n$ be the smallest $\sigma$-field on $I\!N$ which makes $f$ measurable. What is the number of elements in $\mathcal{G}_n$?

[10]

(2) Let $(\Omega, \mathcal{F})$ be a measurable space. Suppose $f, g$ are real valued Borel measurable functions on $(\Omega, \mathcal{F})$. Show that $f + g$ is measurable.

[15]

(3) Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. For $A_1, A_2, \ldots$ in $\mathcal{F}$, show that $\mu(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mu(A_n)$.

[15]

(4) Show that a probability distribution function on $I\!R$ has at most countable number of discontinuities.

[10]

(5) Let $\{Y_n\}_{n\geq 1}$ be a sequence of random variables converging in distribution to a random variable $Y$ as $n \to \infty$. For $n \geq 1$ take $Z_n = Y_n^2 + \frac{1}{n}$. Show that $\{Z_n\}_{n\geq 0}$ converges to $Y^2$ in distribution as $n \to \infty$.

[15]

(6) Let $R, S$ be independent random variables. Suppose $R$ takes values in $\{-1, +1\}$ with $P(R = -1) = P(R = 1) = \frac{1}{2}$ and $S$ has Poisson distribution with parameter $\lambda > 0$, that is, $P(S = n) = e^{-\lambda}\frac{\lambda^n}{n!}$ for $n = 0, 1, 2, \ldots$). Compute characteristic functions of $R + S$ and $R - S$.

[15]

(7) State and prove weak law of large numbers for an i.i.d. sequence of random variables with finite variance.

[20]

(8) (*Bonus question* ) Let $U_1, U_2, \ldots$ be a sequence of i.i.d. random variables with each $U_i$ having uniform distribution in the interval [3, 4]. Show that

$$P\{\omega : \lim_{n\to\infty} U_n(\omega) \text{ exists } \} = 0.$$

[10]